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On The Edge Irregular Reflexive k-Labeling of Some Cartesian Product Graphs

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Abstract. *Let G be a connected and simple graph with vertex set $V(G)$ and edge set $E(G)$. For a graph G we define k-labeling such that the edges of G are labeled with integers $\{1, 2,3, \cdots, k_e\}$ and the vertices of G are labeled with even integers $\{0, 2, 4, \cdots, 2k_v\}$, where $k = max\{k_e, 2k_v\}$. If there is a different weight for all edges, then the labeling is called edge irregular reflexive k-labeling. The weight of edge xy, notated by $wt(xy)$ is defined as a sum of label of x, label of xy and label of y. The minimum k for which G has an edge irregular reflexive k-labeling is defined as reflexive edge strength of G, symbolized by $res(G)$. In this research, we determined the reflexive edge strength of several Cartesian graphs, namely $P_5 \times P_n$, $S_4 \times P_n$, $C_5 \times C_n$, and $F_3 \times P_n$.*

Keywords: *Edge irregular reflexive k-labeling, reflexive edge strength, Cartesian graph.*

Abstrak. *Misalkan G adalah graf sederhana dan terhubung dengan himpunan titik $V(G)$ dan himpunan sisi $E(G)$. untuk graf G kita definisikan pelabelan k sedemikian rupa sehingga sisi-sisi G diberi label dengan bilangan bulat $\{1, 2,3, \cdots, k_e\}$ dan titik-titik G diberi label dengan bilangan bulat genap $\{0, 2, 4, \cdots, 2k_v\}$, dimana $k = max\{k_e, 2k_v\}$. Jika terdapat bobot yang berbeda dengan setiap sisi, maka pelabelan disebut edge irregular reflexive k-labeling. Bobot sisi xy, dinotasikan dengan $wt(xy)$ didefinisikan dengan penjumlahan dari label x, label xy dan label y. k minimum di mana G memiliki pelabelan k-irregular reflexive sisi disebut dengan reflexive edge strength dari graf G, dilambangkan dengan $res(G)$. Pada penelitian ini, kami menentukan reflexive edge strength pada beberapa graf kartesian, yaitu $P_5 \times P_n$, $S_4 \times P_n$, $C_5 \times C_n$, dan $F_3 \times P_n$.*

Kata Kunci: *Pelabelan k-irregular reflexive sisi, reflexive edge strength, graf kartesian.*

1. INTRODUCTION

Discrete mathematics is a branch of mathematics that focuses on discrete or discrete objects, ignoring continuous aspects. Discrete mathematics studies discrete structures such as sets, graphs, combinatorics, logic, number theory, and algorithms [1]. One of the concepts and topics related to discrete mathematics is graph theory [2]. In general, graphs can be used to model various situations in various fields. Including mathematics, computer science, social sciences, communications networks and more [3]. Graphs provide a clear visual representation of how objects are connected and interact with one another [4].

Graph theory studies the structure and properties of graphs. A graph is a pair of sets $G = (V, E)$, where V is the set of vertices and E is the set of edges connecting pairs of vertices. [5]. Topics in graph theory include directed graphs, undirected graphs, weighted graphs, paths, cycles, color mapping and various related algorithms [3]. Graphs can be analyzed and manipulated using various algorithms and techniques, such as shortest path finding algorithms, graph processing algorithms, graph tracing algorithms, and many more. Graphs are also the basis of many data structures used in programming and data analysis.

The labeling of a graph is the mapping of each element in the graph, namely edges, vertices, as well as each edge and vertex with positive integer notation [6]. Each inner element can be labeled, either edge, vertex, or edge and vertex. If every edge of a graph is labeled, it is called edge labeling, if every vertex in a graph is labeled, it is called vertex labeling, if every edge and vertex of a graph is labeled, it is called total labeling [7]. There are several types of labeling on graphs, such as graceful, harmonious, cordial, magic, antimagic, irregular, and others [8]. On this occasion, the labeling used by researchers is irregular labeling because this labeling does not follow structured rules or patterns, or in other words, labels are given freely without any regularity or consistency.

The concept of irregular reflexive labeling is different from the concept of irregular labeling. In k-irregular reflexive labeling, the vertices on a graph are labeled with even non-negative integers, while the edges are labeled with positive integers, and labels of both vertices and edges may be repeated [9]. There are two types of irregular reflexive labeling whose definitions have been introduced, namely edge irregular reflexive k-labeling and vertex irregular reflexive k-labeling. The number of edge irregular reflexive k-labeling of graph G is called reflexive edge strength and denoted by $res(G)$ [10], while the number of vertex irregular reflexive k-labeling of graph G is called reflexive vertex strength and denoted by $rvs(G)$.

For a graph G we define k-labeling such that the edges of G are labeled with integers $\{1, 2, 3, \cdots, k_e\}$ and the vertices of G are labeled with even integers $\{0, 2, 4, \cdots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$[11]. If each edge has a different weight, the labeling is known as edge irregular reflexive k-labeling. The weight of edge xy, denoted by $wt(xy)$, is defined as the sum of the labels of x, xy, and y. The reflexive edge strength of G, denoted by $res(G)$, is defined as the minimum k for which G has an edge irregular reflexive k-labeling. [10].

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Tanna *et. al* have defined $res(G)$, where G is a prism graph (D_n), a wheel graph (W_n), a fan graph (F_n) and a basket (B_n) and defined a lower bound for $res(G)$ [12]. Bača *et. al* also determines $res(G)$, where G is a generalized friendship graph [13]. Bača *et. al* have defined $res(G)$, where G is the cycle graph (C_n), the graph of the cycle Cartesian operations$(C_n x C_3)$, and the results of the joint operations $\left(P_n + (2K1)\right), \left(C_n + (2K1)\right)$ [14]. Guirao *et. al* have determined $res(G)$, where G is a generalized Petersen disjoint union graph [15].

In this research, we started to study the reflexive edge strength of several Cartesian graphs, namely path graphs, star graphs, cycle graphs, and fan graphs. In this research we study Cartesian graph $P_5 \times P_n$ to determine $res(P_5 \times P_n)$, Cartesian graph $S_4 \times P_n$ to determine $res(S_4 \times P_n)$, Cartesian graph $C_5 \times C_n$ to determine $res(C_5 \times C_n)$, and Cartesian graph $F_3 \times P_n$ to determine $res(F_3 \times P_n)$.

2. METHODS

The following method is used to determine the reflexive edge strength of a Cartesian product graph:

a. A graph is defined as a research object.
b. Determine the graph's vertex and edge sets.
c. Using the following Lemma, determine the lower bound of a graph's reflexive edge strength.

$$res(G) \geq \begin{cases} \left\lceil \frac{q}{3} \right\rceil + 1, & if\ q \equiv 2,3(mod\ 6) \\ \left\lceil \frac{q}{3} \right\rceil, & \text{otherwise} \end{cases}$$

where q is the number of edges of graph G.

d. Create vertex and edge labels using the k-irregular reflexive labeling definition.
e. Determine the upper bound of a graph's reflexive edge strength using the obtained function of point 4.
f. If the upper bound of a graph's reflexive edge strength is the same as the lower bound of the graph's reflexive edge strength, then the graph can be used to determine the value of the reflexive edge strength.
g. If the graph's upper bound of reflexive edge strength is different from its lower bound, point 4 is repeated until the upper bound of reflexive edge strength equals the lower bound of reflexive edge strength.

3. RESULTS AND DISCUSSION

Theorem 1. Let $P_5 \times P_n$ be a cartesian product graph with $n \geq 3$, then

$$res(P_5 \times P_n) = \left\lceil \frac{9n-5}{3} \right\rceil.$$

Proof. Let $P_5 \times P_n$ be a cartesian product graph with vertex set $V(P_5 \times P_n) = \{x_i, x_j : 1 \leq i \leq 5, 1 \leq j \leq n\}$ and edge set $E(P_5 \times P_n) = \{x_{ij}x_{(i+1)j}: 1 \leq i \leq 4, 1 \leq j \leq n\} \cup \{x_{ij}x_{i(j+1)}: 1 \leq i \leq 5, 1 \leq j \leq n-1\}$. Cardinality of the vertex set and edge set of $P_5 \times P_n$ are $5n$ and $9n - 5$, respectively. Since $9n - 5 \not\equiv 2,3 (mod\ 6)$, we refer to the lower bound lemma and obtain $res(P_5 \times P_n) \geq \left\lceil \frac{q}{3} \right\rceil = \left\lceil \frac{9n-5}{3} \right\rceil$. Furthermore, to determine the upper bound of $res(P_5 \times P_n)$ by constructing $f_v: V(P_5 \times P_n) \to \{0,2,4,\cdots,2k_v\}$ where $2k_v$ is even number and f_v is surjective function. We also construct the edge labeling, namely $f_e: E(P_5 \times P_n) \to \{1,2,3,\cdots,k_e\}$ where k_e is natural number and f_e is surjective function as follows.

$$f_v(x_{ij}) = \begin{cases} 0, & if\ i = 1,2,3,5 \\ 2, & if\ i = 4,6 \\ 6\left\lceil \frac{i-6}{10} \right\rceil - 2, & if\ i \equiv 0,7,8,9\ \ (mod\ 10) \\ 6\left\lceil \frac{i-6}{10} \right\rceil, & if\ i \equiv 1,2\ \ (mod\ 10) \\ 6\left\lceil \frac{i-6}{10} \right\rceil + 2, & if\ i \equiv 3,4,5,6\ \ (mod\ 10) \end{cases}$$

$$f_e(x_i x_{(i+1)j}) = \begin{cases} 1, & if\ i = 1,3 \\ 2, & if\ i = 2,4 \\ 6\left\lceil \frac{i-4}{8} \right\rceil - 3, & if\ i \equiv 6\ (mod\ 8) \\ 6\left\lceil \frac{i-4}{8} \right\rceil - 2, & if\ i \equiv 5,7\ (mod\ 8) \\ 6\left\lceil \frac{i-4}{8} \right\rceil - 1, & if\ i \equiv 0,3\ (mod\ 8) \\ 6\left\lceil \frac{i-4}{8} \right\rceil, & if\ i \equiv 2,4\ (mod\ 8) \\ 6\left\lceil \frac{i-4}{8} \right\rceil + 1, & if\ i \equiv 1\ (mod\ 8) \end{cases}$$

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$$f_e(x_i x_{i(j+1)}) = \begin{cases} 2, & if\ i = 2,4 \\ 3, & if\ i = 1,3 \\ 5, & if\ i = 5 \\ 3\left\lceil\frac{i-5}{5}\right\rceil + 1, & if\ i \equiv 3\ (mod\ 5) \\ 3\left\lceil\frac{i-5}{5}\right\rceil + 2, & if\ i \equiv 2,4\ (mod\ 5) \\ 3\left\lceil\frac{i-5}{5}\right\rceil + 3, & if\ i \equiv 0,1\ (mod\ 5) \end{cases}$$

Based on the function above, we obtain the consecutive edge weight, namely $wt = \{1,2,3,\cdots,(9n-5)\}$. Since all the edge weights on the $P_5 \times P_n$ graph are different and minimum, then it is an edge irregular reflexive k-labeling. It concludes that $res(P_5 \times P_n) = \left\lceil\frac{9n-5}{3}\right\rceil$.

■

Theorem 2. Let $S_4 \times P_n$ be a cartesian product graph with odd $n \geq 3$, then

$$res(S_4 \times P_n) = \left\lceil\frac{9n-5}{3}\right\rceil.$$

Proof. Let $S_4 \times P_n$ be a cartesian product graph with vertex set $V(S_4 \times P_n) = \{a_j, x_i, x_j : 1 \leq i \leq 4, 1 \leq j \leq n\}$ and edge set $E(S_4 \times P_n) = \{a_j x_{ij} : 1 \leq i \leq 4, 1 \leq j \leq n\} \cup \{a_j a_{j+1} : 1 \leq j \leq n-1\} \cup \{x_{ij} x_{i(j+1)} : 1 \leq i \leq 4, 1 \leq j \leq n-1\}$. Cardinality of the vertex set and edge set of $S_4 \times P_n$ are $5n$ and $9n-5$, respectively. Since $9n-5 \not\equiv 2,3 (mod\ 6)$, we refer to the lower bound lemma and obtain $res(S_4 \times P_n) \geq \left\lceil\frac{q}{3}\right\rceil = \left\lceil\frac{9n-5}{3}\right\rceil$. Furthermore, to determine the upper bound of $res(S_4 \times P_n)$ by constructing $f_v : V(S_4 \times P_n) \rightarrow \{0,2,4,\cdots,2k_v\}$ where $2k_v$ is even number and f_v is surjective function. We also construct the edge labeling, namely $f_e : E(S_4 \times P_n) \rightarrow \{1,2,3,\cdots,k_e\}$ where k_e is natural number and f_e is surjective function as follows.

$$f_v(a_j) = \begin{cases} 0, & if\ i = 1 \\ 12\left\lceil\frac{i-1}{4}\right\rceil - 10, & if\ i \equiv 2\ (mod\ 4) \\ 12\left\lceil\frac{i-1}{4}\right\rceil - 6, & if\ i \equiv 3\ (mod\ 4) \\ 12\left\lceil\frac{i-1}{4}\right\rceil - 4, & if\ i \equiv 0\ (mod\ 4) \\ 12\left\lceil\frac{i-1}{4}\right\rceil, & if\ i \equiv 1\ (mod\ 4) \end{cases}$$

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$$f_v(x_{ij}) = \begin{cases} 0, & if\ i = 1,2 \\ 6\left\lceil\frac{i-2}{8}\right\rceil - 4, & if\ i \equiv 3,4\ (mod\ 8) \\ 6\left\lceil\frac{i-2}{8}\right\rceil - 2, & if\ i \equiv 5,6,7\ (mod\ 8) \\ 6\left\lceil\frac{i-2}{8}\right\rceil, & if\ i \equiv 0,1,2\ (mod\ 8) \end{cases}$$

$$f_e(a_j x_{ij}) = \begin{cases} 6\left\lceil\frac{i}{8}\right\rceil - 5, & if\ i \equiv 1,3\ (mod\ 8) \\ 6\left\lceil\frac{i}{8}\right\rceil - 4, & if\ i \equiv 2,4\ (mod\ 8) \\ 6\left\lceil\frac{i}{8}\right\rceil - 2, & if\ i \equiv 5\ (mod\ 8) \\ 6\left\lceil\frac{i}{8}\right\rceil - 1, & if\ i \equiv 0,6\ (mod\ 8) \\ 6\left\lceil\frac{i}{8}\right\rceil, & if\ i \equiv 7\ (mod\ 8) \end{cases}$$

$$\begin{aligned} f_e(a_j a_{(j+1)}) &= 3 + (i-1)3 \\ &= 3i \end{aligned}$$

$$f_e(x_{ij} x_{i(j+1)}) = \begin{cases} 2, & if\ i = 1,3 \\ 3, & if\ i = 2 \\ 3\left\lceil\frac{i-3}{4}\right\rceil - 2, & if\ i \equiv 0\ (mod\ 4) \\ 3\left\lceil\frac{i-3}{4}\right\rceil + 2, & if\ i \equiv 1,3\ (mod\ 4) \\ 3\left\lceil\frac{i-3}{4}\right\rceil + 3, & if\ i \equiv 2\ (mod\ 4) \end{cases}$$

Based on the function above, we obtain the consecutive edge weight, namely $wt = \{1,2,3, \dots, (9n-5)\}$. Since all the edge weights on the $S_4 \times P_n$ graph are different and minimum, then it is an edge irregular reflexive k-labeling. It concludes that $res(S_4 \times P_n) = \left\lceil\frac{9n-5}{3}\right\rceil$. ■

Theorem 3. Let $C_5 \times C_n$ be a cartesian product graph with $n \geq 3$, then

$$res(C_5 \times C_n) = \begin{cases} \left\lceil\frac{10n}{3}\right\rceil + 1, & if\ 10n \equiv 2,3\ (mod\ 6) \\ \left\lceil\frac{10n}{3}\right\rceil, & otherwise. \end{cases}$$

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Proof. Let $C_5 \times C_n$ be a cartesian product graph with vertex set $V(C_5 \times C_n) = \{x_{ij} : 1 \le i \le 5, 1 \le j \le n\}$ and edge set $E(C_5 \times C_n) = \{x_{ij}x_{(i+1)j} : 1 \le i \le 4, 1 \le j \le n\} \cup \{x_{1j}x_{5j} : 1 \le j \le n\} \cup \{x_{ij}x_{i(j+1)} : 1 \le i \le 5, 1 \le j \le n-1\} \cup \{x_{i1}x_{in} : 1 \le i \le 5\}$.

Cardinality of the vertex set and edge set of $C_5 \times C_n$ are $5n$ and $10n$, respectively. Since $10n \equiv 2,3 (mod\ 6)$, we refer to the lower bound lemma and obtain $res(C_5 \times C_n) \ge \left\lceil \frac{q}{3} \right\rceil + 1 = \left\lceil \frac{10n}{3} \right\rceil + 1$. Furthermore, to determine the upper bound of $res(C_5 \times C_n)$ by constructing $f_v : V(C_5 \times C_n) \to \{0,2,4,\cdots,2k_v\}$ where $2k_v$ is even number and f_v is surjective function. We also construct the edge labeling, namely $f_e : E(C_5 \times C_n) \to \{1,2,3,\cdots,k_e\}$ where k_e is natural number and f_e is surjective function as follows.

$$f_v(x_{ij}) = \begin{cases} 0, & if\ i = 1,2,3,4 \\ 2, & if\ i = 5 \\ 20\left\lceil \frac{i-5}{30} \right\rceil - 16, & if\ i \equiv 8\ (mod\ 30) \\ 20\left\lceil \frac{i-5}{30} \right\rceil - 14, & if\ i \equiv 6,7,9,10,13\ (mod\ 30) \\ 20\left\lceil \frac{i-5}{30} \right\rceil - 12, & if\ i \equiv 11,12,14,15\ (mod\ 30) \\ 20\left\lceil \frac{i-5}{30} \right\rceil - 10, & if\ i \equiv 18\ (mod\ 30) \\ 20\left\lceil \frac{i-5}{30} \right\rceil - 8, & if\ i \equiv 17,19\ (mod\ 30) \\ 20\left\lceil \frac{i-5}{30} \right\rceil - 6, & if\ i \equiv 16,20,22,23,24\ (mod\ 30) \\ 20\left\lceil \frac{i-5}{30} \right\rceil - 4, & if\ i \equiv 21,25\ (mod\ 30) \\ 20\left\lceil \frac{i-5}{30} \right\rceil - 2, & if\ i \equiv 28\ (mod\ 30) \\ 20\left\lceil \frac{i-5}{30} \right\rceil, & if\ i \equiv 0,3,26,27,29\ (mod\ 30) \\ 20\left\lceil \frac{i-5}{30} \right\rceil + 2, & if\ i \equiv 1,2,4,5\ (mod\ 30) \end{cases}$$

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$$f_e(x_{ij}x_{(i+1)j}) = \begin{cases} 3, & if\ i = 1 \\ 1, & if\ i = 2 \\ 2, & if\ i = 3,4 \\ 20\left\lceil \frac{i-4}{24} \right\rceil - 14, & if\ i \equiv\ 5,6\ (mod\ 24) \\ 20\left\lceil \frac{i-4}{24} \right\rceil - 13, & if\ i \equiv\ 7,8,9,10\ (mod\ 24) \\ 20\left\lceil \frac{i-4}{24} \right\rceil - 12, & if\ i \equiv\ 11,12\ (mod\ 24) \\ 20\left\lceil \frac{i-4}{24} \right\rceil - 8, & if\ i \equiv\ 13\ (mod\ 24) \\ 20\left\lceil \frac{i-4}{24} \right\rceil - 7, & if\ i \equiv\ 15,17,18\ (mod\ 24) \\ 20\left\lceil \frac{i-4}{24} \right\rceil - 6, & if\ i \equiv\ 14,19,20\ (mod\ 24) \\ 20\left\lceil \frac{i-4}{24} \right\rceil - 5, & if\ i \equiv\ 16\ (mod\ 24) \\ 20\left\lceil \frac{i-4}{24} \right\rceil - 2, & if\ i \equiv\ 21,22\ (mod\ 24) \\ 20\left\lceil \frac{i-4}{24} \right\rceil - 1, & if\ i \equiv\ 0,1,2,23\ (mod\ 24) \\ 20\left\lceil \frac{i-4}{24} \right\rceil, & if\ i \equiv\ 3,4\ (mod\ 24) \end{cases}$$

$$f_e(x_{1j}x_{5j}) = \begin{cases} 3, & if\ i = 1 \\ 20\left\lceil \frac{i-1}{6} \right\rceil - 12, & if\ i \equiv 2\ (mod\ 6) \\ 20\left\lceil \frac{i-1}{6} \right\rceil - 11, & if\ i \equiv 3\ (mod\ 6) \\ 20\left\lceil \frac{i-1}{6} \right\rceil - 8, & if\ i \equiv 4\ (mod\ 6) \\ 20\left\lceil \frac{i-1}{6} \right\rceil - 7, & if\ i \equiv 5\ (mod\ 6) \\ 20\left\lceil \frac{i-1}{6} \right\rceil, & if\ i \equiv 0\ (mod\ 6) \\ 20\left\lceil \frac{i-1}{6} \right\rceil + 1, & if\ i \equiv 1\ (mod\ 6) \end{cases}$$

$$f_e(x_{ij}x_{i(j+1)}) = \begin{cases} 3, & if\ i = 1 \\ 1, & if\ i = 2 \\ 2, & if\ i = 3,4,5 \\ 6, & if\ i = 6 \\ 4, & if\ i = 7 \\ 5, & if\ i = 8,9,10 \\ 20\left\lceil\frac{i-10}{30}\right\rceil - 11, & if\ i \equiv 11,12\ (mod\ 30) \\ 20\left\lceil\frac{i-10}{30}\right\rceil - 10, & if\ i \equiv 14,15,16,17\ (mod\ 30) \\ 20\left\lceil\frac{i-10}{30}\right\rceil - 9, & if\ i \equiv\ 18,19,20\ (mod\ 30) \\ 20\left\lceil\frac{i-10}{30}\right\rceil - 8, & if\ i \equiv\ 13(mod\ 30) \\ 20\left\lceil\frac{i-10}{30}\right\rceil - 5, & if\ i \equiv 21,22\ (mod\ 30) \\ 20\left\lceil\frac{i-10}{30}\right\rceil - 4, & if\ i \equiv 24,25,26,27\ (mod\ 30) \\ 20\left\lceil\frac{i-10}{30}\right\rceil - 3, & if\ i \equiv\ 0,28,29\ (mod\ 30) \\ 20\left\lceil\frac{i-10}{30}\right\rceil - 2, & if\ i \equiv\ 23\ (mod\ 30) \\ 20\left\lceil\frac{i-10}{30}\right\rceil + 1, & if\ i \equiv 2\ (mod\ 30) \\ 20\left\lceil\frac{i-10}{30}\right\rceil + 2, & if\ i \equiv 4,7\ (mod\ 30) \\ 20\left\lceil\frac{i-10}{30}\right\rceil + 3, & if\ i \equiv 1,9\ (mod\ 30) \\ 20\left\lceil\frac{i-10}{30}\right\rceil + 4, & if\ i \equiv 3,5,6\ (mod\ 30) \\ 20\left\lceil\frac{i-10}{30}\right\rceil + 5, & if\ i \equiv 8,10\ (mod\ 30) \end{cases}$$

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$$f_e(x_{ij}x_{i(j-1)}) = \begin{cases} 20\left\lceil \frac{i}{30} \right\rceil - 7, & if\ i \equiv\ 2\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil - 6, & if\ i \equiv\ 4\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil - 5, & if\ i \equiv\ 1\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil - 4, & if\ i \equiv\ 3,5\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil - 1, & if\ i \equiv\ 11\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil, & if\ i \equiv\ 15\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil + 1, & if\ i \equiv\ 12\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil + 2, & if\ i \equiv\ 6,7,13,14\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil + 3, & if\ i \equiv\ 9,10\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil + 4, & if\ i \equiv\ 23\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil + 5, & if\ i \equiv\ 8,22\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil + 6, & if\ i \equiv\ 24\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil + 7, & if\ i \equiv\ 21\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil + 8, & if\ i \equiv\ 16,17,23,25\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil + 9, & if\ i \equiv\ 18,19,20\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil + 14, & if\ i \equiv\ 27\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil + 15, & if\ i \equiv\ 29\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil + 16, & if\ i \equiv\ 26\ (mod\ 30) \\ 20\left\lceil \frac{i}{30} \right\rceil + 17, & if\ i \equiv\ 0,28\ (mod\ 30) \end{cases}$$

Based on the function above, we obtain the consecutive edge weight, namely $wt = \{1,2,3,\cdots,10n\}$. Since all the edge weights on the $C_5 \times C_n$ graph are different and minimum, then it is an edge irregular reflexive k-labeling. It concludes that $res(C_5 \times C_n) = \left\lceil \frac{10n}{3} \right\rceil + 1$, for $10n \equiv 2,3\ (mod\ 6)$ and $\left\lceil \frac{10n}{3} \right\rceil$ for otherwise.

■

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Theorem 4. Let $F_3 \times P_n$ be a cartesian product graph with $n \geq 2$, then

$$res(F_3 \times P_n) = \begin{cases} \left\lceil \frac{9n-4}{3} \right\rceil + 1, & if\ n \equiv 0,2\ (mod\ 4) \\ \left\lceil \frac{9n-4}{3} \right\rceil, & otherwise. \end{cases}$$

Proof. Let $F_3 \times P_n$ be a cartesian product graph with vertex set $V(F_3 \times P_n) = \{a_{j,}x_i, x_j : 1 \leq i \leq 3, 1 \leq j \leq n\}$ and edge set $E(F_3 \times P_n) = \{a_j x_{ij}: 1 \leq i \leq 3, 1 \leq j \leq n\} \cup \{x_{ij}x_{ij}: 1 \leq i \leq 3, 1 \leq j \leq n\} \cup \{a_j a_{j+1}: 1 \leq j \leq n\} \cup \{x_{ij}x_{i(j+1)}: 1 \leq i \leq 3, 1 \leq j \leq n\}$. Cardinality of the vertex set and edge set of $F_3 \times P_n$ are $4n$ and $(9n-4)$, respectively. For $n \equiv 0\ (mod\ 4)$, we obtain $9n - 4 = 9(4a) - 4 = 36a - 4$. Since $36a - 4 \equiv 2\ (mod\ 6)$, we refer to the lower bound lemma and obtain $res(F_3 \times P_n) \geq \left\lceil \frac{q}{3} \right\rceil + 1 = \left\lceil \frac{9n-4}{3} \right\rceil + 1$. For $n \equiv 2\ (mod\ 4)$, we obtain $9n - 4 = 9(4a + 2) - 4 = 36a + 18 - 4 = 36a + 14$. Since $36a + 14 \equiv 2\ (mod\ 6)$, we refer to the lower bound lemma and obtain $res(F_3 \times P_n) \geq \left\lceil \frac{q}{3} \right\rceil + 1 = \left\lceil \frac{qn-4}{3} \right\rceil + 1$. Furthermore, to determine the upper bound of $res(F_3 \times P_n)$ by constructing $f_v: V(F_3 \times P_n) \rightarrow \{0,2,4,\cdots,2k_v\}$ where $2k_v$ is even number and f_v surjective function. We also construct the edge labeling, namely $f_e: E(F_3 \times P_n) \rightarrow \{1,2,3,\cdots,k_e\}$ where k_e is natural number and f_e is surjective function as follows.

$$f_v(a_j) = \begin{cases} 0, & if\ i = 1 \\ 12\left\lceil \frac{i-1}{4} \right\rceil - 8, & if\ i \equiv 2\ (mod\ 4) \\ 12\left\lceil \frac{i-1}{4} \right\rceil - 6, & if\ i \equiv 3\ (mod\ 4) \\ 12\left\lceil \frac{i-1}{4} \right\rceil - 2, & if\ i \equiv 0\ (mod\ 4) \\ 12\left\lceil \frac{i-1}{4} \right\rceil, & if\ i \equiv 1\ (mod\ 4) \end{cases}$$

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$$f_v(x_{1j}) = \begin{cases} 0, & if\ i = 1 \\ 12\left\lceil\frac{i-1}{4}\right\rceil - 8, & if\ i \equiv 2\ (mod\ 4) \\ 12\left\lceil\frac{i-1}{4}\right\rceil - 6, & if\ i \equiv 3\ (mod\ 4) \\ 12\left\lceil\frac{i-1}{4}\right\rceil - 2, & if\ i \equiv 0\ (mod\ 4) \\ 12\left\lceil\frac{i-1}{4}\right\rceil, & if\ i \equiv 1\ (mod\ 4) \end{cases}$$

$$f_v(x_{2j}) = \begin{cases} 0, & if\ i = 1 \\ 12\left\lceil\frac{i-1}{4}\right\rceil - 8, & if\ i \equiv 2\ (mod\ 4) \\ 12\left\lceil\frac{i-1}{4}\right\rceil - 4, & if\ i \equiv 3\ (mod\ 4) \\ 12\left\lceil\frac{i-1}{4}\right\rceil - 2, & if\ i \equiv 0\ (mod\ 4) \\ 12\left\lceil\frac{i-1}{4}\right\rceil + 2, & if\ i \equiv 1\ (mod\ 4) \end{cases}$$

$$f_v(x_{3j}) = \begin{cases} 12\left\lceil\frac{i}{4}\right\rceil - 10, & if\ i \equiv 1\ (mod\ 4) \\ 12\left\lceil\frac{i}{4}\right\rceil - 8, & if\ i \equiv 2\ (mod\ 4) \\ 12\left\lceil\frac{i}{4}\right\rceil - 4, & if\ i \equiv 3\ (mod\ 4) \\ 12\left\lceil\frac{i}{4}\right\rceil - 2, & if\ i \equiv 0\ (mod\ 4) \end{cases}$$

$$f_e(a_j x_{1j}) = \begin{cases} 12\left\lceil\frac{i}{4}\right\rceil - 11, & if\ i \equiv 1\ (mod\ 4) \\ 12\left\lceil\frac{i}{4}\right\rceil - 10, & if\ i \equiv 2\ (mod\ 4) \\ 12\left\lceil\frac{i}{4}\right\rceil - 5, & if\ i \equiv 3\ (mod\ 4) \\ 12\left\lceil\frac{i}{4}\right\rceil - 4, & if\ i \equiv 0\ (mod\ 4) \end{cases}$$

$$f_e(a_j x_{2j}) = \begin{cases} 2, i = 1 \\ 3 + (i-1)3 = 3i \end{cases}$$

$$f_e(a_j x_{3j}) = \{2 + (i-1)3 = 3i - 1$$

$$f_e(x_{1j} x_{2j}) = \begin{cases} 3, i = 1 \\ 4 + (i-1)3 = 3i + 1 \end{cases}$$

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$$f_e(x_{2j}x_{3j}) = \begin{cases} 3, & if\ i = 1 \\ 12\left\lceil\frac{i-1}{4}\right\rceil - 6, & if\ i \equiv 2\ (mod\ 4) \\ 12\left\lceil\frac{i-1}{4}\right\rceil - 5, & if\ i \equiv 3\ (mod\ 4) \\ 12\left\lceil\frac{i-1}{4}\right\rceil, & if\ i \equiv 0\ (mod\ 4) \\ 12\left\lceil\frac{i-1}{4}\right\rceil + 1, & if\ i \equiv 1\ (mod\ 4) \end{cases}$$

$$f_e(a_j a_{j+1}) = \{2 + (i-1)3 = 3i - 1$$

$$f_e(x_{1j}x_{1(j+1)}) = \{3 + (i-1)3 = 3i$$

$$f_e(x_{2j}x_{2(j+1)}) = \begin{cases} 4, i = 1 \\ 5 + (i = 1)3 = 3i + 2 \end{cases}$$

$$f_e(x_{3j}x_{3(j+1)}) = \{3 + (i-1)3 = 3i$$

Based on the function above, we obtain the consecutive edge weight, namely $wt = \{1,2,3,\dots,(9n-4)\}$. Since all the edge weights on the $F_3 \times P_n$ graph are different and minimum, then it is an edge irregular reflexive k-labeling. It concludes that $res(F_3 \times P_n) = \left\lceil\frac{9n-4}{3}\right\rceil + 1$ for $n \equiv 0,2\ (mod\ 4)$ and $\left\lceil\frac{9n-4}{3}\right\rceil$ for otherwise. ■

An ilustration of Edge Irregular Reflexive Labeling on $P_5 \times P_5$ is given by figure 1.

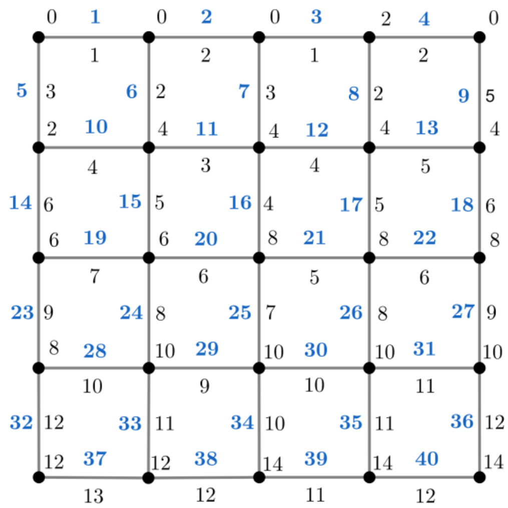

Figure 1 An Illustration of Edge Irregular Reflexive Labeling on $P_5 \times P_5$

Figure 1 illustrates the irregular reflexive labeling edges on graph $P_5 \times P_5$. The point labels on the graph $P_5 \times P_5$ are $\{0,2,4,6,8,10,12,14\}$, while the edge labels are $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$, as shown in the figure. The largest label obtained based on the labeling results is 14. This is in accordance with the research findings in Theorem 1.

4. CONCLUSSION

This study generates a theorem through reflexive k-irregular labeling on path, star, cycle, and fan graphs based on the results and discussion. In this study, we analyze the cartesian graph $P_5 \times P_n$ to determine $res(P_5 \times P_n)$, cartesian graph $S_4 \times P_n$ to determine $res(S_4 \times P_n)$, cartesian graph $C_5 \times C_n$ to determine $res(C_5 \times C_n)$, and cartesian graph $F_3 \times P_n$ to determine $res(F_3 \times P_n)$. It is proposed that future research investigate the res values on graphs other than those provided in this paper as well as search for their existence characteristics.

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