SUPER EDGE ANTIMAGIC TOTAL LABELING ON DISJOINT UNION OF CYCLE WITH CHORD

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Abstract

A graph $G$ with order $n$ and size $m$ is called $(a,d)$-edge antimagic total $((a,d)$-EAT) if there exist integers $a \leq 0$, $d \geq 0$ and a bijection $\alpha : V \cup E \rightarrow \{1, 2, 3, \ldots, n + m\}$ such that $W = \{w(uv), uv \in E\} = \{a, a + d, a + (q - 1)\}$, where $w(uv) = \alpha(u) + \alpha(v) + \alpha(uv)$. An $(a,d)$-EAT labeling $\alpha$ of graph $G$ is super if $\alpha(V) = \{1, 2, \ldots, n\}$. In this paper we describe how to construct a super $(a,d)$-EAT labeling on some classes of disjoint union from non-isomorphic graphs, namely disjoint union of cycle with cycle with chord $\mu \cup \mu - 1 \mu$. 

Keywords: $(a,d)$-edge-antimagic total labeling, super $(a,d)$-edge antimagic total labeling, cycle with chord, $\mu \cup \mu - 1 \mu$.

1. Introduction

All graphs in this paper are finite, undirected, and simple. $V(G)$ and $E(G)$ (in short, $V$ and $E$) stand for the vertex-set and edge-set of graph $G$, respectively. Let $e = \{u, v\}$ (in short, $e = uv$) denote an edge connecting vertices $u$ and $v$ in $G$. Then, let order $|V(G)|$ in $G$ denoted by $p$ and size $|E(G)|$ in $G$ denoted by $q$.

A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers (usually to the positive integers). If the domain is the vertex-set or the edge-set, the labelings are called respectively vertex-labelings or edge-labelings. In this paper we deal with the case where the domain is $V \cup E$, and these are called total labeling. We define the edge-weight of an edge $uv \in E$ under a total labeling to be the sum of the vertex labels corresponding to vertices $u$, $v$ and edge label corresponding to edge $uv$. General references for graph-theoretic notions is [12]. A general survey of graph labelings is [6].

A graph $G$ is called $(a,d)$-edge antimagic total $((a,d)\text{-EAT})$ if there exist integers $a \geq 0$, $d \geq 0$ and a bijection function $f : V \cup E \rightarrow \{1, 2, \ldots, p + q\}$ such that the set of edge-weights is $w(uv) = f(u) + f(uv) + f(v)$, $uv \in E$, form an arithmetic progression $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$. In particular, an $(a,d)$-EAT labeling of graph $G$ is super if $f : V \rightarrow \{1, 2, \ldots, p\}$. Thus, a super $(a,d)$-edge-antimagic total graph is a graph that admits a $(a,d)$-edge-antimagic total labeling.

The concept of $(a,d)$-edge antimagic total labeling, introduced by Simanjuntak et al. in [14], is natural extension of the notion of edge-magic labeling defined by Kotzig and Rosa.
The super \((a,d)\)-edge-antimagic total labeling is a natural extension of the notion of super edge-magic labeling which was defined by Enomoto et al. in [4]. In this paper we investigate the existence of super \((a,d)\)-edge-antimagic total labelings for disjoint union of non isomorphic graphs. A number of classification studies on super \((a,d)\)-EAT (resp. \((a,d)\)-EAT) for disjoint union of non isomorphic graphs has been extensively investigated. For instances, some constructions of super \((a,0)\)-edge-antimagic total labelings for \(nG_k \cup mP_k\) and \(K_{1,m} \cup K_{1,n}\) have been shown by Ivančo and Lučkaničová in [7]. In [5] Sudarsana, Ismailmuza, Baskoro, and Assiyatun show that \(nP_2 \cup P_{n+2}\), \(nP_2 \cup P_{n+2}\), and \(nP_2 \cup P_{n+2}\) are super edge antimagic total labeling. Dafik et al also found disjoint union of non isomorphic graph which admits super \((a,d)\)-edge-antimagic total labelings, namely \(mK_{1,m} \cup S_{k,1}\) in [2].

More results concerning on super edge antimagic total labeling, see for instances in a nice survey paper by Gallian [6].

Now, we will concentrate on the disjoint union of cycle with cycle with chord, denoted by \(mC_2^k \cup (\mu - 1)C_n \cup \mu C_n^3 \cup mC_n\).

2. Some Useful Lemmas

We start this section by a necessary condition for a graph to be super \((a, d)\)-edge-antimagic total, providing a least upper bound for feasible values of \(d\).

**Lemma 1** If a \((p,q)\)-graph is super \((a,d)\)-edge-antimagic total then \(d \leq \frac{2p+q-5}{q-1}\).

**Proof.** Assume that a \((p,q)\)-graph has a super \((a,d)\)-edge-antimagic total labeling \(f : V(G) \cup E(G) \to \{1,2,\ldots,p+q\}\). The minimum possible edge-weight in the labeling \(f\) is at least \(1 + 2 + p + 1 = p + 4\). Thus, \(a \geq p + 4\). On the other hand, the maximum possible edge-weight is at most \((p - 1) + p + (p + q) = 3p + q - 1\). So we obtain \(a + (q - 1)d \leq 3p + q - 1\) which gives the desired upper bound for the difference \(d\).

The following lemma, proved by Figueroa-Centeno et al. in [13], gives a necessary and sufficient condition for a graph to be super edge magic (super \((a,0)\)-edge antimagic total).

**Lemma 2** A \((p,q)\)-graph \(G\) is super edge-magic if and only if there exists a bijective function \(f : V(G) \to \{1,2,\ldots,p\}\) such that the set \(S = \{f(u) + f(v) : uv \in E(G)\}\) consists of \(q\) consecutive integers. In such a case, \(f\) extends to a super edge-magic labeling of \(G\) with magic constant \(a = p + q + s\), where \(s = \min(S)\) and \(S = \{a - (p + 1), a - (p + 2), \ldots, a - (p + q)\}\).
In our terminology, the previous lemma states that a \((p, q)\)-graph \(G\) is super \((a, 0)\)-edge antimagic total if and only if there exists an \((a - p - q, 1)\)-edge antimagic vertex labeling.

Next, we restate the following lemma that appeared in [8].

**Lemma 3** [8] Let \(\mathcal{K}\) be a sequence \(\mathcal{K} = \{c, c + 1, c + 2, \ldots, c + k\}\), \(k\) even. Then there exists a permutation \(\Pi(\mathcal{K})\) of the elements of \(\mathcal{K}\) such that \(\mathcal{K} + \Pi(\mathcal{K}) = \left\{2c + \frac{k}{2}, 2c + \frac{k}{2} + 1, 2c + \frac{k}{2} + 2, \ldots, 2c + \frac{3k}{2} - 1, 2c + \frac{3k}{2}\right\}\)

### 3. Disjoint Union of Cycle with Cycle With Chord

We shall write \(C^k\) to mean the graph constructed from a cycle \(C_n\) by joining two vertices whose distance in the cycle is \(k\) [10]. Now, we will study super edge-antimagic of a disjoint union of cycle graph with cycle with chord, denoted by \(C^k \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n\), for \(\mu \geq 1, (\mu - 1) \leq m \leq \mu, n \geq 7\) and \(n\) odd. It is a disconnected graph with vertex set \(V = \{x_i | 1 \leq i \leq n, 1 \leq j \leq 2m + 2\mu - 1\}\) and edge set:

\[E = \{x_i x_{i+1} \cup x_j x_{j+1} | 1 \leq i \leq n, 1 \leq j \leq 2m + 2\mu - 1\} \cup \{x_j x_{j+1} | 1 \leq j \leq m\} \cup \{x_i x_{i+1} | (m + \mu) \leq j \leq m + 2\mu - 1\}\].

Thus : \(p = m(n) + (\mu - 1)(n) + \mu(n) + m(n) = n(2m + 2\mu - 1)\)
and \(q = m(n + 1) + (\mu - 1)(n) + \mu(n + 1) + m(n) = n(2m + 2\mu - 1) + m + \mu\).

If the disjoint union of \(mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n\), has a super \((a, d)\)-edge antimagic total labeling then, for \(p = n(2m + 2\mu - 1)\) and \(q = n(2m + 2\mu - 1) + m + \mu\), it follows from Lemma 1 that the upper bound of \(d\) is \(d \leq 3 - \left(\frac{2m + 2\mu + 2}{2m + 2\mu + 2 + n + m + \mu - 1}\right)\), \(d \geq 0\), \(d\) is integer, so \(d \in \{0, 1, 2\}\).

The following theorem describes an \((a, 1)\)-edge antimagic vertex labeling for disjoint union of \(mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n\), for \(\mu \geq 1, (\mu - 1) \leq m \leq \mu, n \geq 7\) and \(n\) odd.

**Lemma 4** The disjoint union of \(mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n\) has an \((a, 1)\)-edge antimagic vertex labeling for \(\mu \geq 1, (\mu - 1) \leq m \leq \mu, n \geq 7\) and \(n\) odd.

**Proof.** Define the vertex labeling \(\alpha_i : V(mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n) \rightarrow \{1, 2, \ldots, n(2m + 2\mu - 1)\}\) in the following way:

For \(x_i \in V\), \(1 \leq i \leq n - 1, 1 \leq j \leq m + \mu\)
\[ \alpha_1(x^j_i) = \frac{i(2m+2\mu-1)+2j+1}{2} + \frac{(-1)^{i+1}}{4} n(2m + 2\mu - 1) + (-1)^{i}(m + \mu) - \frac{(-1)^{i+1}}{2} 3j + \frac{(-1)^{i+1}}{2} \]

For \( x^j_i \); \( 1 \leq i \leq n - 1 \), \( m + \mu + 1 \leq j \leq 2m + 2\mu - 1 \)
\[ \alpha_1(x^j_i) = \frac{(2m+2\mu-1) - 2(m+\mu) + 2j + 1}{2} + \frac{(-1)^{i+1}}{4} n(2m + 2\mu - 1) + \frac{(-1)^{i}}{2} 2(m + \mu) - \frac{(-1)^{i+1}}{2} 3j \]

For \( x^j_n \); \( 1 \leq j \leq m + \mu \)
\[ \alpha_1(x^j_n) = \frac{i(2m+2\mu-1)+2j+1}{2} \]

For \( x^j_n \); \( m + \mu + 1 \leq j \leq 2m + 2\mu - 1 \)
\[ \alpha_1(x^j_n) = \frac{(2m+2\mu-1)-4(m+\mu)+2j+1}{2} \]

The vertex labeling \( \alpha_1 \) is a bijective function. The edge weights of \( mC^k_n \cup (\mu - 1)C_n \cup \muC^3_n \cup mC_n \), under the labeling \( \alpha_1 \), constitute the following sets:

For \( x^j_i x^j_{i+1} \); \( 1 \leq i \leq n - 1 \), \( 1 \leq j \leq 2m + 2\mu - 1 \)
\[ w^1_{\alpha_1}(x^j_i x^j_{i+1}) = \frac{(n+2i)(2m+2\mu-1)+2(m+\mu)-2j+3}{2}, \quad 1 \leq i \leq n - 2; \quad 1 \leq j \leq m + \mu \]
\[ w^2_{\alpha_1}(x^j_i x^j_{i+1}) = \frac{(n+2i)(2m+2\mu-1)+6(m+\mu)-2j+1}{2}, \quad 1 \leq i \leq n - 2; \quad m + \mu + 1 \leq j \leq 2m + 2\mu - 1 \]
\[ w^3_{\alpha_1}(x^j_i x^j_{i+1}) = \frac{3n(2m+2\mu-1)-2j+3}{2}, \quad i = n - 1; \quad 1 \leq j \leq 2m + 2\mu - 1 \]

For \( x^j_n x^j_1 \); \( 1 \leq j \leq 2m + 2\mu - 1 \)
\[ w^4_{\alpha_1}(x^j_n x^j_1) = \frac{n(2m+2\mu-1)+4j-1}{2}, \quad 1 \leq j \leq m + \mu \]
\[ w^5_{\alpha_1}(x^j_n x^j_1) = \frac{n(2m+2\mu-1)-4(m+\mu)+4j+1}{2}, \quad m + \mu + 1 \leq j \leq 2m + 2\mu - 1 \]

For \( x^j_3 x^j_{n-2} \); \( 1 \leq j \leq m + \mu \)
\[ w^6_{\alpha_1}(x^j_3 x^j_{n-2}) = \frac{n(2m+2\mu-1)-2(m+\mu)+4j+1}{2} \]

For \( x^j_1 x^j_{n-2} \); \( m + \mu \leq j \leq m + 2\mu - 1 \)
\[ w^7_{\alpha_1}(x^j_1 x^j_{n-2}) = \frac{n(2m+2\mu-1)-6(m+\mu)+4j+3}{2} \]

It is not difficult to see that the set:
\[ \bigcup_{i=1}^{n} w^7_{\alpha_3} = \left\{ \frac{n(2m+2\mu-1)-2(m+\mu)+3}{2}, \frac{n(2m+2\mu-1)-2(m+\mu)+5}{2}, \ldots, \frac{3n(2m+2\mu-1)+1}{2} \right\} \]
consists of consecutive integers. Thus \( \alpha_1 \) is an \( \left( \frac{n(2m+2\mu-1)-2(m+\mu)+3}{2}, 1 \right) \)-edge antimagic vertex labeling. □

**Theorem 1** The disjoint union of \( mC^k_n \cup (\mu - 1)C_n \) has a super \( \left( \frac{5n(2m+2\mu-1)+3}{2}, 0 \right) \)-edge antimagic total labeling and a super \( \left( \frac{3n(2m+2\mu-1)-2(m+\mu)+5}{2}, 2 \right) \)-edge antimagic total labeling, for \( \mu \geq 1, (\mu - 1) \leq m \leq \mu, n \geq 7 \) and \( n \) odd.

**Proof. Case 1.** for \( d = 0 \)

We have proved that the vertex labeling \( \alpha_1 \) is a \( \left( \frac{n(2m+2\mu-1)-2(m+\mu)+3}{2}, 1 \right) \)-edge antimagic vertex labeling. With respect to Lemma 2, by completing the edge labels \( p + 1, p + 2, \ldots, p + q \), we are able to extend labeling \( \alpha_1 \) to a super \( (a, 0) \)-edge antimagic total labeling, where, for \( p = n(2m + 2\mu - 1) \) and \( q = n(2m + 2\mu - 1) + m + \mu \), the value \( a = \frac{5n(2m+2\mu-1)+3}{2} \) □

**Proof. Case 2.** for \( d = 2 \)

Label the vertices of \( mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n \) with \( \alpha_2(x_i^j) = \alpha_1(x_i^j) \) and \( \alpha_2(x_n^j) = \alpha_1(x_n^j) \), for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, 2m + 2\mu - 1 \); and label the edges with the following way:

For \( x_i^j x_{i+1}^j \), \( 1 \leq i \leq n - 1, 1 \leq j \leq 2m + 2\mu - 1 \)
\[ \alpha_2 = (n + i)(2m + 2\mu - 1) + 2(m + \mu) - j + 1, \quad 1 \leq i \leq n - 2, \quad 1 \leq j \leq m + \mu \]
\[ \alpha_2 = (n + i)(2m + 2\mu - 1) + 4(m + \mu) - j, \quad 1 \leq i \leq n - 2, \quad m + \mu + 1 \leq j \leq 2m + 2\mu - 1 \]
\[ \alpha_2 = 2n(2m + 2\mu - 1) + (m + \mu) - j + 1, \quad i = n - 1, \quad 1 \leq j \leq 2m + 2\mu - 1 \]

For \( x_n^j x_1^j \), \( 1 \leq j \leq 2m + 2\mu - 1 \)
\[ \alpha_2 = n(2m + 2\mu - 1) + (m + \mu) + 2j - 1, \quad 1 \leq j \leq m + \mu \]
\[ \alpha_2 = n(2m + 2\mu - 1) - (m + \mu) + 2j, \quad m + \mu + 1 \leq j \leq 2m + 2\mu - 1 \]

For \( x_3^j x_{n-2}^j \), \( 1 \leq j \leq m \)
\[ \alpha_2 = n(2m + 2\mu - 1) + 2j \]

For \( x_1^j x_{n-2}^j \), \( m + \mu \leq j \leq m + 2\mu - 1 \)
\[ \alpha_2 = n(2m + 2\mu - 1) - 2(m + \mu) + 2j + 1 \]

The total labeling \( \alpha_2 \) is a bijective function from \( V(mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n) \cup E(mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n) \) onto the set \{1, 2, 3, \ldots, 2n(2m + 2\mu - 1) + m + \mu\}. The edge-weights of \( mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n \), under the labeling \( \alpha_2 \), constitute the sets:

For \( x_i^j x_{i+1}^j \), \( 1 \leq i \leq n - 1, 1 \leq j \leq 2m + 2\mu - 1 \)
\[ W_{\alpha_1}^1 = \{ w_{\alpha_1}^i + \alpha_2(x_i^j x_{i+1}^j), \ jka \ 1 \leq i \leq n - 2; \ 1 \leq j \leq m + \mu \} \]
\[ = \left( \frac{(n+2i)(2m+2\mu-1+2(m+\mu)-j+3)}{2} + (n + i)(2m + 2\mu - 1) + 2(m + \mu) - j + 1 \right) \]
\[ = \left( \frac{(3n+4i)(2m+2\mu-1)+6(m+\mu)-4j+5}{2} \right) \]
\[ W_{\alpha_2}^2 = \{ w_{\alpha_2}^i + \alpha_2(x_i^j x_{i+1}^j), \ jka \ 1 \leq i \leq n - 2; \ m + \mu + 1 \leq j \leq 2m + 2\mu - 1 \} \]
\[ = \left( \frac{(n+2i)(2m+2\mu-1+6(m+\mu)-2j+1)}{2} + (n + i)(2m + 2\mu - 1) + 4(m + \mu) - j \right) \]
\[ = \left( \frac{(3n+4i)(2m+2\mu-1)+14(m+\mu)-4j+1}{2} \right) \]
\[ W_{\alpha_2}^3 = \{ w_{\alpha_2}^i + \alpha_2(x_i^j x_{i+1}^j), \ jka \ i = n - 2; \ 1 \leq j \leq 2m + 2\mu - 1 \} \]
\[ = \left( \frac{3n(2m+2\mu-1)-2j+3}{2} + 2n(2m + 2\mu - 1) + (m + \mu) - j + 1 \right) \]
\[ = \left( \frac{7n(2m+2\mu-1)+2(m+\mu)-4j+5}{2} \right) \]

For \( x_n^j x_{n-1}^j, \ 1 \leq j \leq 2m + 2\mu - 1 \)
\[ W_{\alpha_1}^4 = \{ w_{\alpha_1}^i + \alpha_2(x_n^j x_{n-1}^j), \ jka \ 1 \leq j \leq m + \mu \} \]
\[ = \left( \frac{n(2m+2\mu-1)+4j-1}{2} + n(2m + 2\mu - 1) + (m + \mu) + 2j - 1 \right) \]
\[ = \left( \frac{3n(2m+2\mu-1)+2(m+\mu)+8j-3}{2} \right) \]
\[ W_{\alpha_2}^5 = \{ w_{\alpha_2}^i + \alpha_2(x_n^j x_{n-1}^j), \ jka \ m + \mu + 1 \leq j \leq 2m + 2\mu - 1 \} \]
\[ = \left( \frac{n(2m+2\mu-1)-4(m+\mu)+4j+1}{2} + n(2m + 2\mu - 1) - (m + \mu) + 2j \right) \]
\[ = \left( \frac{3n(2m+2\mu-1)-6(m+\mu)+8j+1}{2} \right) \]

For \( x_n^j x_{n-2}^j, \ 1 \leq j \leq m \)
\[ W_{\alpha_2}^6 = \{ w_{\alpha_2}^i + \alpha_2(x_n^j x_{n-2}^j) \} \]
\[ = \left( \frac{n(2m+2\mu-1)-2(m+\mu)+4j+1}{2} + n(2m + 2\mu - 1) + 2j \right) \]
\[ = \left( \frac{3n(2m+2\mu-1)-2(m+\mu)+8j+1}{2} \right) \]

For \( x_n^j x_{n-2}, \ m + \mu \leq j \leq m + 2\mu - 1 \)
\[ W_{\alpha_2}^7 = \{ w_{\alpha_2}^i + \alpha_2(x_n^j x_{n-2}^j) \} \]
\[ = \left( \frac{n(2m+2\mu-1)-6(m+\mu)+4j+3}{2} + n(2m + 2\mu - 1) - 2(m + \mu) + 2j + 1 \right) \]
\[ = \left( \frac{3n(2m+2\mu-1)-10(m+\mu)+8j+5}{2} \right) \]

It is not difficult to see that the set of:
\[ U_{i=1}^{7} W_{\alpha_1}^i = \left\{ \frac{3n(2m+2\mu-1)-2(m+\mu)+5}{2}, \frac{n(2m+2\mu-1)-2(m+\mu)+9}{2}, \ldots, \frac{7n(2m+2\mu-1)+2(m+\mu)+1}{2} \right\}, \]
contains an arithmetic sequence with the first term \(\frac{3n(2m+2\mu-1)-2(m+\mu)+5}{2}\) and common difference 2. Thus \(\alpha_2\) is a super \(\frac{3n(2m+2\mu-1)-2(m+\mu)+5}{2}\)-edge antimagic total labeling. 

This concludes the proof. \(\square\)

**Theorem 2** The disjoint union of \(mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n\) has a super \(\left(\frac{4n(2m+2\mu-1)-(m+\mu)+4}{2}, 1\right)\)-edge antimagic total labeling for \(\mu \geq 1, (\mu - 1) \leq m \leq \mu, n \geq 7\) and \(n\) odd.

**Proof.** We will prove using Lemma 3. For \(\mu \geq 1, (\mu - 1) \leq m \leq \mu, n \geq 7\) and \(n\) odd, consider the vertex labeling \(\alpha_1\) of the graph \(mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n\) from Lemma 4 which is a \(\left(\frac{n(2m+2\mu-1)-2(m+\mu)+3}{2}, 1\right)\)-EAV labeling. Let sequence \(\mathcal{N} = \{c, c + 1, c + 2, \ldots, c + k\}\) be the set of edge-weights of the vertex labeling \(\alpha_1\) for \(c = \frac{n(2m+2\mu-1)-2(m+\mu)+3}{2}\) and \(k = n(2m + 2\mu - 1) - (m + \mu) - 1\). In light of Lemma 3, there exists a permutation \([\prod(\mathcal{N})]\) of the elements of \(\mathcal{N}\) such that \(\mathcal{N} + \left[\prod(\mathcal{N}) + \frac{k}{2} + \frac{m+\mu}{2}\right] = \left\{2c + k + \frac{m+\mu}{2}, 2c + k + \frac{m+\mu}{2} + 1, \ldots, 2c + 2k + \frac{m+\mu}{2}\right\}\). If \(\left[\prod(\mathcal{N}) + \frac{k}{2} + \frac{m+\mu}{2}\right]\) is an edge labeling of \(mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n\), then \(\mathcal{N} + \left[\prod(\mathcal{N}) + \frac{k}{2} + \frac{m+\mu}{2}\right]\) gives the set of the edge-weights of \(mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n\), which implies that the resulting total labeling is super \(\left(\frac{4n(2m+2\mu-1)-(m+\mu)+4}{2}, 1\right)\)-EAT. This concludes the proof. \(\square\)

4. Conclusion

We have lemma and theorem from bijective function of super \((a, d)\)-edge antimagic total labeling on disjoint union of cycle with chord:

- **Lemma 4** The disjoint union of \(mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n\) has an \((a, 1)\)-edge antimagic vertex labeling for \(\mu \geq 1, (\mu - 1) \leq m \leq \mu, n \geq 7\) and \(n\) odd.

- **Theorem 1** The disjoint union of \(mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n\) has a super \(\left(\frac{5n(2m+2\mu-1)+3}{2}, 0\right)\)-edge antimagic total labeling and a super \(\left(\frac{3n(2m+2\mu-1)-2(m+\mu)+5}{2}, 2\right)\)-edge antimagic total labeling, for \(\mu \geq 1, (\mu - 1) \leq m \leq \mu, n \geq 7\) and \(n\) odd.

- **Theorem 4** The disjoint union of \(mC^k_n \cup (\mu - 1)C_n \cup \mu C^3_n \cup mC_n\) has a super \(\left(\frac{4n(2m+2\mu-1)-(m+\mu)+4}{2}, 1\right)\)-edge antimagic total labeling for \(\mu \geq 1, (\mu - 1) \leq m \leq \mu, n \geq 7\) and \(n\) odd.
5. References


